Noether's theory for non-conservative generalised mechanical systems

1980 J. Phys. A: Math. Gen. 13 431

(http://iopscience.iop.org/0305-4470/13/2/011)

J. Phys. A: Math. Gen., **13** (1980) 431–435. Printed in Great Britain

Noether's theory for non-conservative generalised mechanical systems

Dj S Djukic† and A M Strauss

Department of Engineering Science, University of Cincinnati, Cincinnati, Ohio 45221, USA

Received 15 June 1979

Abstract. Noether's theorem and Noether's inverse theorem for generalised mechanical systems described by Lagrangian functions of the second order and non-conservative forces are established. The existence of the first integral depends on the existence of solutions of the generalised Noether–Bessel–Hagen equation. The theory is based on the idea that the transformations of time and generalised coordinates together with non-conservative forces determine the transformations of velocities and accelerations. An illustrative problem is discussed.

1. Introduction

Problems in the calculus of variations whose Lagrange function involves higher-order derivatives have received considerable attention ever since the origins of the subject in the early eighteenth century. The interest in second-order problems lies in the fact that the corresponding results can be applied to different branches of physics. Such problems enjoy considerable attention in relativity and continuum mechanics. Also, at the present time, efforts have been made to establish a 'generalised mechanics' and a 'generalised electrodynamics' by including higher-order derivatives in the Lagrangian.

Higher-order variation problems are studied in most textbooks on the variational calculus. Advanced monographs have been written by Grässer (1967) and Logan (1977). Conservation laws for higher-order problems were originally considered by Noether (1918), while Anderson (1973) puts the theory in more modern form. Conservation laws for second-order variational problems, whose corresponding Euler equations are of fourth order, are discussed by Blakeslee and Logan (1976) and Logan (1977).

Here, we will concentrate our attention on the conservation laws for a system of ordinary differential equations of fourth order, but where the existence of the corresponding variational principle is not imposed. It will yield a significant extension of the previous results. Our theory is based on Noether's theory for classical non-conservative mechanics (see Djukic and Vujanovic 1975, Vujanovic 1978).

In this paper the following conventions will be observed: (1) the summation convention is employed throughout; (2) lower case italic indices imply a range of values from 1 to n. The paper also uses the terminology of generalised mechanics.

† On leave from the University of Novi Sad, 21000 Novi Sad, Yugoslavia.

Let us consider a generalised mechanical system with n degrees of freedom, where the q^i are regarded as the generalised coordinates and t is time. The conservative part of the system can be described completely by a Lagrangian function of second order, $L(t, q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n, \ddot{q}^1, \ldots, \ddot{q}^n) \equiv L(t, q, \dot{q}, \ddot{q})$, where $\dot{} \equiv \mathrm{d}/\mathrm{d}t$ and $\ddot{} \equiv \mathrm{d}^2/\mathrm{d}t^2$. Let the corresponding differential equations of motion be

$$L_{(i)} \equiv \frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial \ddot{q}^i} + Q_i(t, q, \dot{q}, \ddot{q}, \dddot{q}) \doteq 0 \tag{1}$$

where the symbol $\doteq$ is employed to represent an equality holding only 'along the orbit', i.e., for $q(t)$ satisfying the equations of motion. Here, the generalised non-conservative forces Q_i are functions of the variables indicated. The governing equations of motion (1) form a system of fourth-order ordinary differential equations. The corresponding conservation laws in the works of Noether (1918), Anderson (1973), Blakeslee and Logan (1976) and Logan (1977) are obtained for the case when the system does not contain non-conservative forces, that is when $Q_i \equiv 0$.

2. Noether's theorem

Let us consider a continuous one-parameter transformation of the generalised coordinates, generalised velocities, generalised accelerations and time of the form

$$\bar{t} \approx t + \epsilon\psi(t, q, \dot{q}, \ddot{q}) \tag{2}$$

$$\bar{q}^i \approx q^i + \epsilon\phi^i(t, q, \dot{q}, \ddot{q}) \tag{3}$$

$$\mathrm{d}\bar{q}^i/\mathrm{d}\bar{t} \approx \dot{q}^i + \epsilon[\dot{\phi}^i - \dot{q}^i\dot{\psi} + \Phi^i(t, q, \dot{q}, \ddot{q}, \dddot{q})] \tag{4}$$

$$\mathrm{d}^2\bar{q}^i/\mathrm{d}\bar{t}^2 \approx \ddot{q}^i + \epsilon[\ddot{\phi}^i - 2\ddot{q}^i\dot{\psi} - \dot{q}^i\ddot{\psi} + \dot{\Phi}^i + \omega^i(t, q, \dot{q}, \ddot{q}, \dddot{q}, \ddddot{q})] \tag{5}$$

where ϵ is a small parameter of the transformation and ψ, ϕ^i, Φ^i and ω^i are functions of the corresponding variables. In the existing forms of Noether's theory for the system (1), where Q_i are equal to zero, the functions Φ^i and ω^i are identically equal to zero, that is the transformations of velocities and acceleration are known completely by the transformations of time and coordinates. Here, through the quantities Φ^i and ω^i, the effect of non-conservative forces will be introduced. Thus, corresponding to (2)–(5) there exists an infinitesimal transformation of the form

$$\Delta t \approx \epsilon\psi \qquad \Delta q^i \approx \epsilon\phi^i \qquad \Delta\dot{q}^i \approx \epsilon(\dot{\phi}^i - \dot{q}^i\dot{\psi} + \Phi^i) \tag{6}$$

$$\Delta\ddot{q}^i \approx \epsilon(\ddot{\phi}^i - 2\ddot{q}^i\dot{\psi} - \dot{q}^i\ddot{\psi} + \dot{\Phi}^i + \omega^i). \tag{7}$$

Further, let us assume that the infinitesimal transformation (6) and (7) induces a function $\mathscr{H}$, given by

$$\mathrm{d}\mathscr{H} = L(t, q, \dot{q}, \ddot{q})\,\mathrm{d}t, \tag{8}$$

to be gauge variant, i.e. to be 'invariant up to an exact differential' in the sense that

$$\Delta(\mathrm{d}\mathscr{H}) \equiv L(\bar{t}, \bar{q}, \mathrm{d}\bar{q}/\mathrm{d}\bar{t}, \mathrm{d}^2\bar{q}/\bar{\mathrm{d}}t^2)\,\mathrm{d}\bar{t} - L(t, q, \dot{q}, \ddot{q})\,\mathrm{d}t = \epsilon\,\mathrm{d}\Lambda(t, q, \dot{q}, \ddot{q}, \dddot{q}) \tag{9}$$

where Λ is a known function of t, q, $\dot{q}$, $\ddot{q}$ and $\dddot{q}$. Now, combining (2), (6), (7) and (9), developing the term $L(\bar{t}, \ldots, \mathrm{d}^2\bar{q}/\bar{\mathrm{d}}t^2)$ in series and retaining only members linear in the small parameter ϵ, the expression (9) becomes

$$\frac{\Delta(\mathrm{d}\mathscr{H})}{\mathrm{d}t}=\epsilon\Big(L\dot{\psi}+\psi\dot{L}+\frac{\partial L}{\partial q^i}\Omega^i+\frac{\partial L}{\partial \dot{q}^i}(\dot{\Omega}^i+\Phi^i)+\frac{\partial L}{\partial \ddot{q}^i}(\ddot{\Omega}^i+\dot{\Phi}^i+\omega^i)\Big) \tag{10}$$

where

$$\Omega^i=\phi^i-\dot{q}^i\psi. \tag{11}$$

From (10), after simple manipulation and use of (1), we have

$$\frac{\Delta(\mathrm{d}\mathscr{H})}{\mathrm{d}t}=\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\Big[L\psi+\Big(\frac{\partial L}{\partial \dot{q}^i}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}^i}\Big)\Omega^i+\frac{\partial L}{\partial \ddot{q}^i}\dot{\Omega}^i\Big]$$
$$+\epsilon\Big(\frac{\partial L}{\partial \dot{q}^i}\Phi^i+\frac{\partial L}{\partial \ddot{q}^i}(\dot{\Phi}^i+\omega^i)-Q_i\Omega^i\Big)+\epsilon\Omega^iL_{(i)}. \tag{12}$$

Assuming that the functions Φ^i, ω^i and Ω^i satisfy the equation

$$\frac{\partial L}{\partial \dot{q}^i}\Phi^i+\frac{\partial L}{\partial \ddot{q}^i}(\dot{\Phi}^i+\omega^i)=Q_i\Omega^i, \tag{13}$$

and remembering that the generalised mechanical system under consideration moves in agreement with the equations (1), we may deduce the following theorem from (9) and (12).

Theorem I (*Noether*). If the expression (8) is gauge variant in the sense of equation (9) under the one-parameter infinitesimal transformation (6) and (7), which satisfies equation (13), then the quantity

$$D(t,q,\dot{q},\ddot{q},\dddot{q})=L\psi+\Big(\frac{\partial L}{\partial \dot{q}^i}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}^i}\Big)\Omega^i+\frac{\partial L}{\partial \ddot{q}^i}\dot{\Omega}^i-\Lambda \tag{14}$$

is a first integral (D = constant) of the equations of motion (1). Further, combining (9), (10), (11) and (13) we have the necessary condition for the functions ψ and Ω^i, which must be satisfied if the expression (8) is gauge variant under the infinitesimal transformations (6) and (7):

$$L\dot{\psi}+\psi\dot{L}+\frac{\partial L}{\partial q^i}\Omega^i+\frac{\partial L}{\partial \dot{q}^i}\dot{\Omega}^i+\frac{\partial L}{\partial \ddot{q}^i}\ddot{\Omega}^i+Q_i\Omega^i=\dot{\Lambda} \tag{15}$$

while the functions Φ^i and ω^i must satisfy equation (13). The equations (13) and (15) may be called the generalised Noether–Bessel–Hagen equations (see Djukic and Vujanovic 1975) for non-conservative generalised mechanical systems. For the case of a conservative mechanical system, i.e. when $Q_i=0$, the equations (14) and (15) constitute the classical form of Noetherian theory (see, for example, Logan 1977, pp 117–24). In practical applications, if the equation (15) admits a solution for ψ and Ω^i, then a conserved quantity (14) automatically exists. Here we may remark that after integration of equation (15) with respect to ψ and Ω^i the functions Φ^i and ω^i can be easily found from equation (13) (for example, choosing the functions Φ^i and after that solving the corresponding algebraic equation for ω^i).

Remembering (see (2), (3) and (11)) that the functions ψ and Ω^i do not depend on the $\ddot{q}^i$'s and $\dddot{q}^i$'s and developing explicitly the time derivatives $\dot{\psi}$ and $\dot{\Omega}^i$ (for example, $\dot{\psi}=\partial\psi/\partial t+\dot{q}^i\,\partial\psi/\partial q^i+\ddot{q}^i\,\partial\psi/\partial\dot{q}^i+\dddot{q}^i\,\partial\psi/\partial\ddot{q}^i$) we can split equation (15) into a system of linear partial differential equations with respect to ψ and Ω^i (for this the procedure from Djukic and Vujanovic (1975) and Vujanovic (1978) must be generalised). The equations may have two forms. The first form will be obtained if we consider the $\dddot{q}^i$'s as the independent variables in (15). The second form will be established if we express the $\dddot{q}^i$'s in terms of t, q^i, $\dot{q}^i$, $\ddot{q}^i$ and $\dddot{q}^i$, by using the differential equations of motion (1), and substitute these relations in (15) (see Vujanovic 1978). In both cases the question of the integrability conditions for these systems of partial differential equations is raised immediately.

3. Noether's inverse theorem

Using the explicit form of the equations of motion (1), the time dependence of a given arbitrary quantity $G(t, q, \dot{q}, \ddot{q}, \dddot{q})$ may be expressed as

$$\frac{\mathrm{d}G}{\mathrm{d}t}=f_G+J^{ki}\frac{\partial G}{\partial \dddot{q}^i}L_{(i)} \tag{16}$$

where

$$\begin{aligned} f_G=&\frac{\partial G}{\partial t}+\frac{\partial G}{\partial q^i}\dot{q}^i+\frac{\partial G}{\partial \dot{q}^i}\ddot{q}^i+\frac{\partial G}{\partial \ddot{q}^i}\dddot{q}^i\\ &-J^{ki}\frac{\partial G}{\partial \dddot{q}^i}\Big[\frac{\partial L}{\partial q^i}+Q_i-\frac{\partial^2 L}{\partial t\,\partial \dot{q}^i}-\frac{\partial^2 L}{\partial \dot{q}^i\,\partial \dot{q}^\delta}\ddot{q}^\delta-\frac{\partial^2 L}{\partial \dot{q}^i\,\partial q^\delta}\dot{q}^\delta\\ &-\frac{\partial^2 L}{\partial \dot{q}^i\,\partial \ddot{q}^\delta}\dddot{q}^\delta+\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial^2 L}{\partial \ddot{q}^i\,\partial t}+\frac{\partial^2 L}{\partial \ddot{q}^i\,\partial q^\delta}\dot{q}^\delta+\frac{\partial^2 L}{\partial \ddot{q}^i\,\partial \dot{q}^\delta}\ddot{q}^\delta\Big)+\dddot{q}^\delta\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial^2 L}{\partial \ddot{q}^i\,\partial \ddot{q}^\delta}\Big]. \end{aligned} \tag{17}$$

J is the inverse matrix to the matrix H of the Lagrangian

$$H_{ij}=\frac{\partial^2 L}{\partial \ddot{q}^i\,\partial \ddot{q}^\delta} \qquad H\neq 0 \qquad H_{ij}J^{ki}=\delta_j^k \tag{18}$$

and δ_j^k is the Kronecker delta symbol. Combining (16) and (12) we have the identity

$$\begin{aligned} \epsilon\frac{\mathrm{d}G}{\mathrm{d}t}=&\epsilon f_G-\frac{\Delta(\mathrm{d}\mathscr{H})}{\mathrm{d}t}+\epsilon L_{(i)}\Big(J^{ki}\frac{\partial G}{\partial \dddot{q}^i}+\Omega^i\Big)+\epsilon\Big(\frac{\partial L}{\partial \dot{q}^i}\Phi^i+\frac{\partial L}{\partial \ddot{q}^i}(\dot{\Phi}^i+\omega^i)-Q_i\Omega^i\Big)\\ &+\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\Big[L\psi+\Big(\frac{\partial L}{\partial \dot{q}^i}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}^i}\Big)\Omega^i+\frac{\partial L}{\partial \ddot{q}^i}\dot{\Omega}^i\Big], \end{aligned} \tag{19}$$

which must hold for any choice of the infinitesimal transformation (6) and (7). In the special case when

$$L\psi+\Big(\frac{\partial L}{\partial \dot{q}^i}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}^i}\Big)\Omega^i+\frac{\partial L}{\partial \ddot{q}^i}\dot{\Omega}^i-\Lambda-G=C_1 \tag{20}$$

$$\frac{\partial L}{\partial \dot{q}^i}\Phi^i+\frac{\partial L}{\partial \ddot{q}^i}(\dot{\Phi}^i+\omega^i)=Q_i\Omega^i, \tag{21}$$

where C_1 is an arbitrary constant, from (19) we have

$$\epsilon f_G=\frac{\Delta(\mathrm{d}\mathscr{H})}{\mathrm{d}t}-\epsilon\dot{\Lambda}-\epsilon L_{(i)}\Big(\Omega^i+J^{ki}\frac{\partial G}{\partial \dddot{q}^i}\Big), \tag{22}$$

and therefore (16) can be written in the form

$$\epsilon \frac{\mathrm{d}G}{\mathrm{d}t} \doteq \frac{\Delta(\mathrm{d}\mathscr{H})}{\mathrm{d}t} - \epsilon \dot{\Lambda}. \tag{23}$$

When we apply this equation to a constant of motion G, that is, when $\mathrm{d}G/\mathrm{d}t = 0$, we obtain $\Delta(\mathrm{d}\mathscr{H}) = \epsilon\, \mathrm{d}\Lambda$. Hence we have the following theorem.

Theorem II (*Noether's inverse*). To every constant of motion G for the generalised mechanical system described by the Lagrangian L and non-conservative forces Q_i, there corresponds an infinitesimal transformation satisfying (20) and (21) that leaves the expression (8) gauge variant in the sense of equation (9).

Remark. The infinitesimal transformations which satisfy (20) and (21) are not unique. For example, one solution to (20) is

$$\Omega^i = \frac{\partial L}{\partial \ddot{q}^i} \qquad \psi = L^{-1}\left(C_1 + \Lambda + G - \frac{\partial L}{\partial \dot{q}^i} \frac{\partial L}{\partial \ddot{q}^i}\right) \tag{24}$$

while for the solutions of (21) the previous discussion after theorem I is valid.

4. An example

To illustrate the present theory, let us consider a generalised mechanical system with one degree of freedom ($q \equiv x$), whose Lagrangian function and non-conservative force are

$$L = \tfrac{1}{2}\ddot{x}^2 + \tfrac{1}{2}a\dot{x}^2 + \tfrac{1}{2}bx^2 \qquad Q = \mu\dot{x} + (\mu^2/a^2)\ddot{x} - (2\mu/a)\dddot{x} \tag{25}$$

where a, b and μ are arbitrary constants. Assuming the function Λ to be of the form

$$\Lambda = \mathrm{e}^{-\mu t/a}[(2\mu/a)\dot{x}\ddot{x} - (\mu^2/a^2)x\ddot{x}], \tag{26}$$

we have a solution to the Noether–Bessel–Hagen equation (15) as

$$\psi = \mathrm{e}^{-\mu t/a} \qquad \Omega = \mathrm{e}^{-\mu t/a}[(\mu/2a)x - \dot{x}]. \tag{27}$$

Substituting (25)–(27) into (14) we obtain a conserved quantity for the system

$$D = \tfrac{1}{2}\,\mathrm{e}^{-\mu t/a}\{bx^2 - a\dot{x}^2 - \ddot{x}^2 + \mu x[\dot{x} + (\mu/a^2)\ddot{x}] + 2\dot{x}\dddot{x} - (\mu/a)(x\dddot{x} + \dot{x}\ddot{x})\}. \tag{28}$$

References

Anderson D 1973 *J. Phys. A: Math., Nucl. Gen.* **6** 299–305
Blakeslee J S and Logan J D 1976 *Nuovo Cim.* B **34** 319–24
Djukic Dj S and Vujanovic D B 1975 *Acta Mech.* **23** 18–27
Grässer H S P 1967 *University of South Africa Mathematics Communication M2*
Logan J D 1977 *Invariant Variational Principles* (New York: Academic)
Noether E 1918 *Nach R. Akad. Wiss. Gottingen, Math. Phys.* Kl II 235–57
Vujanovic B D 1978 *Int. J. Non-linear Mech.* **13** 185–97